Two Families Theorem: A Few Combinatorial Applications

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RAA 2017, St. Petersburg, May 22–26, 2017

Outline

- What are Two Families Theorems

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  - *Combinatorial Version*
  - *Algebraic Versions*

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  $\cdots$ *and if time permits* $\cdots$
- My interest in these objects

Two Families Theorems

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Towards the first statement

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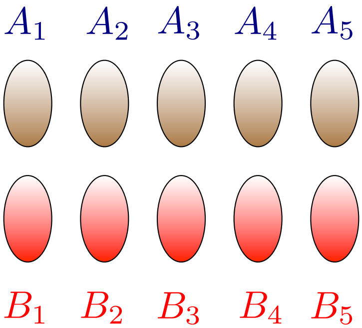
Two Families Theorem: An Illustration



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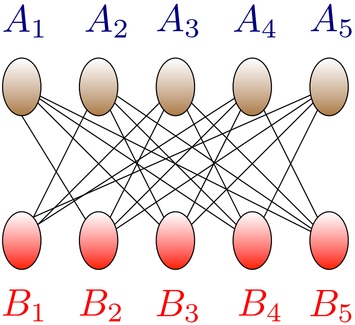

Draw an edge between two sets if the intersect!

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- *An upper bound that is independent of n — the universe size?*

Two Families Theorem (Sets): Bollabás Theorem

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Then $m \leqslant \binom{p+q}{p}$.

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- This implies that in this case $m = \binom{p+q}{p}$!

Two Families Theorem (Sets)
A Weaker Proof

Proof – Slightly Weaker Upper Bound

- We call a partition (X, Y) of the universe U *good* for a pair (A_i, B_i) if $A_i \subseteq X$ and $B_i \subseteq Y$.

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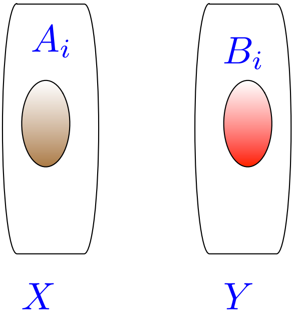

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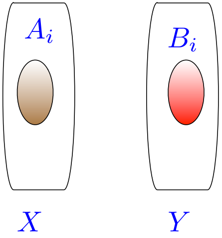

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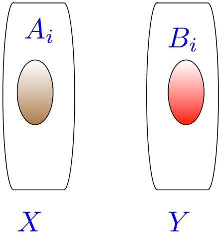

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Is it possible that the partition (X, Y) *could be good for some other pair* (A_j, B_j) *where* $i \neq j$*?*

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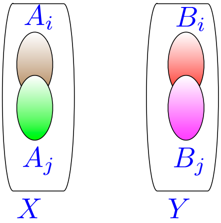

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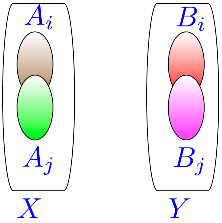

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But this would imply that $A_i \cap B_j = \varnothing$ and $A_j \cap B_i = \varnothing$ – a contradiction!

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For every pair (A_i, B_i) — define

$$\mathcal{P}_i = \{(X, Y) \mid (X, Y) \text{ is good for } (A_i, B_i)\}.$$

Essentially, a set containing all the partitions of U that are good for the pair (A_i, B_i).

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Thus,

$$m2^{n-(p+q)} \leqslant \sum_{i=1}^{m} |\mathcal{P}_i| \leqslant 2^n \implies m \leqslant 2^{p+q}!$$

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$$\begin{aligned}
\Pr[X_i] &= \frac{\binom{n}{p+q} p!q!(n-(p+q))!}{n!} \\
&= \frac{\frac{n!}{(n-(p+q))!(p+q)!} p!q!(n-(p+q))!}{n!} \\
&= \frac{p!q!}{(p+q)!} \\
&= \frac{1}{\binom{p+q}{p}}
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- CLAIM: X_i's are *pairwise disjoint events*.
- Let Π be an order in which *all the elements of A_i precede all those of B_i in this order* and *all the elements of A_j precede all those of B_j in this order*.
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- $m \leqslant \binom{p+q}{p}$.

Two Families Theorem (Sets)
Application I

Transversals

- Let U be a universe. For a collection of sets $\mathcal{F} \subseteq 2^U$, we call $T \subseteq U$ a *transversal* of $\mathcal{F}$, if for all $A \in \mathcal{F}$; $A \cap T \neq \emptyset$.

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- Denote the size of the smallest transversal by $\tau(\mathcal{F})$.

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- This is a τ-critical system of size $\binom{p+q}{p}$, where $\tau(\mathcal{F}) = q+1$ and $\forall A \in \mathcal{F};\ |A| = p$.

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- However, B_i does not intersect A_i, otherwise it would also be a transversal of $\mathcal{F}$.
- Bollabás Theorem: $m = |\mathcal{F}| \leqslant \binom{p+q}{p}$.

Two Families Theorem (Sets)
Application II

Vertex Separators

A vertex subset S of a graph G is a vertex separator for non-adjacent vertices s and t if removal of S from the graph separates s and t into distinct connected components. In other words, in $G - S$ there is no path from s to t.

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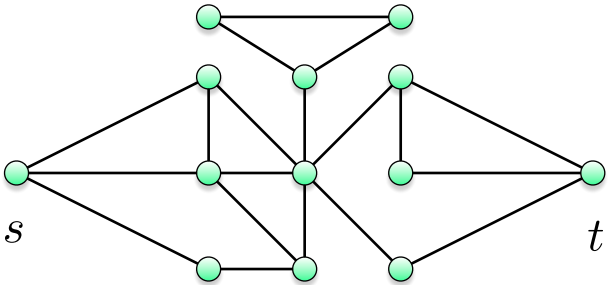

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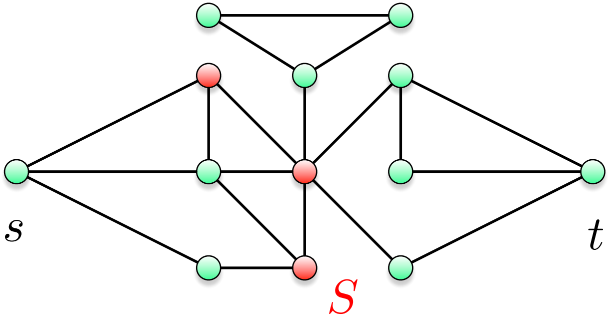

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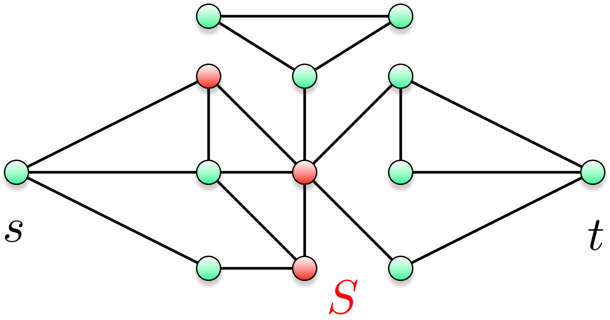

Is S shown above a minimal (s, t)-separator?.

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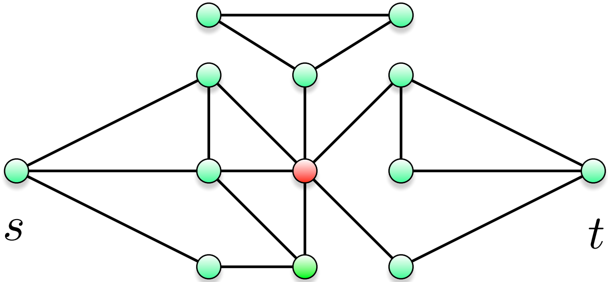

Characterization of Minimal Separators

A (s, t)-vertex separator S in G is minimal if and only if the graph $G - S$, obtained by removing S from G, has two connected components A_S containing s and B_S containing t such that each vertex in S is both adjacent to some vertex in A_S and to some vertex in B_S.

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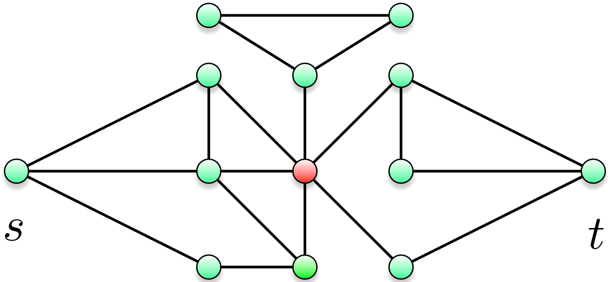

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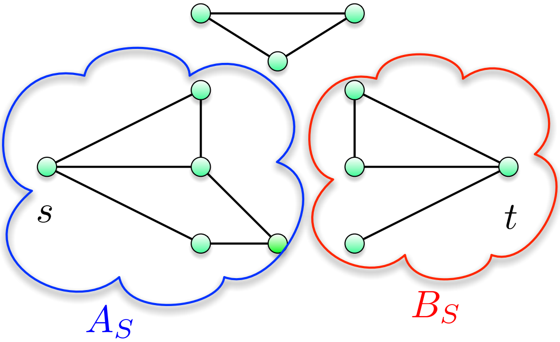

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Can we prove something better?

- Let $F(p,q)^{st}$ denote the set of minimal (s,t)-vertex separators S such that $|A_S| = p$ and $|S| = q$.

$$F(p,q)^{st} = \{S \mid |S| = q \bigwedge S \text{ a minimal } (s,t)\text{-vertex separator} \bigwedge |A_S| = p\}$$

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Clearly,

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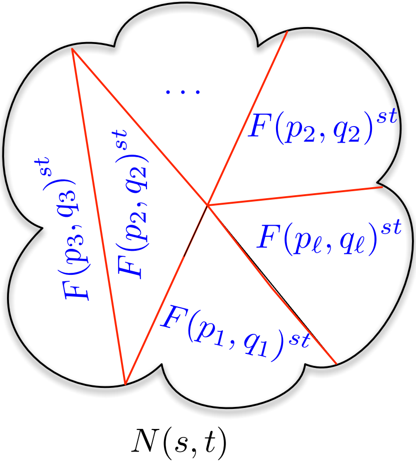

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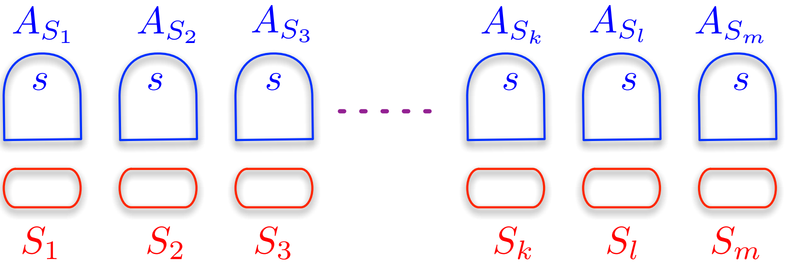
Bounding $F(p,q)^{st}$



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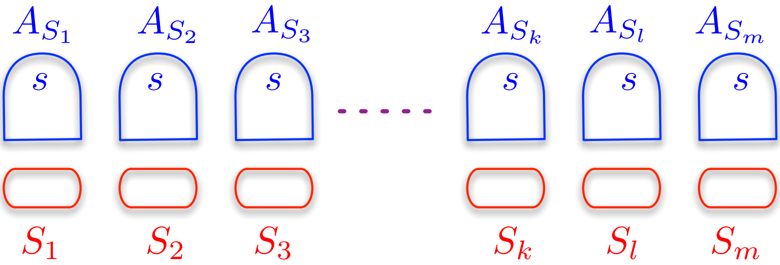

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Notice that this is true about any p, q. However, let us put $p = n/2$ and $q = n/2$ and we get $m = |F(p,q)^{st}| \leqslant \binom{p+q}{p} \sim 2^n$.

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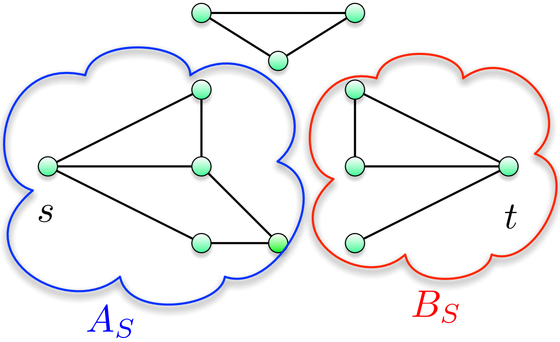
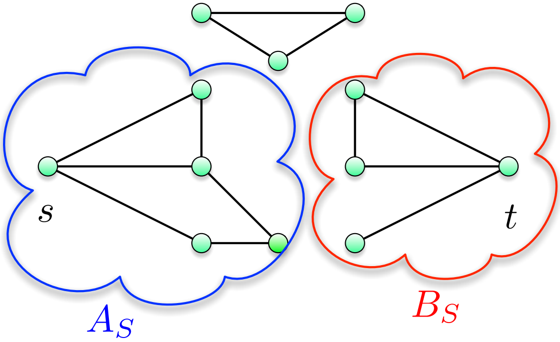


For every minimal separator S, we have that either $|A_S| \leqslant \frac{n-|S|}{2}$ or $|B_S| \leqslant \frac{n-|S|}{2}$



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Thus to get an upper bound we only need to bound those separators for which we have that $2p+q \leqslant n$.

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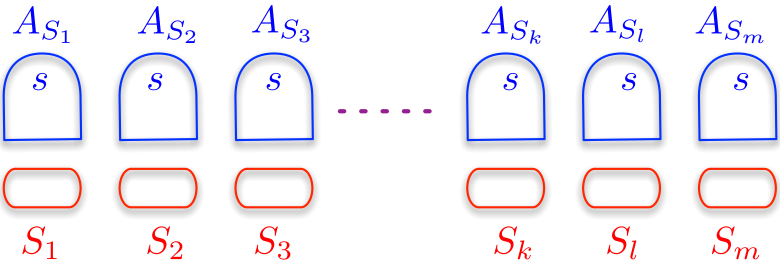

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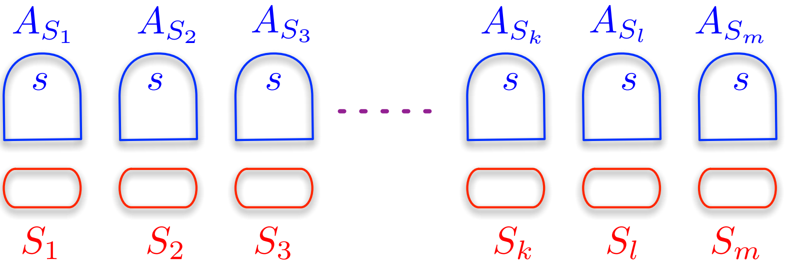

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Notice that this is true about any p, q for which $2p + q \leqslant n$ and for any s, t. Thus, the number of minimal (s,t)-vertex separators in a graph is at most $1.618^n n^{\mathcal{O}(1)}$.

Open Problem

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- Consequences — Improved exact exponential time algorithms for computing TREEWIDTH, finding induced subgraph of constant treewidth (like finding MINIMUM FEEDBACK VERTEX SET), ...

Two Families Theorem
Subspaces

Two Families Theorem: Subspaces

Let $A_1, \ldots, A_m$ be p dimensional and $B_1, \ldots, B_m$ be q dimensional subspaces of a vector space W over a field $\mathbb{F}$ such that

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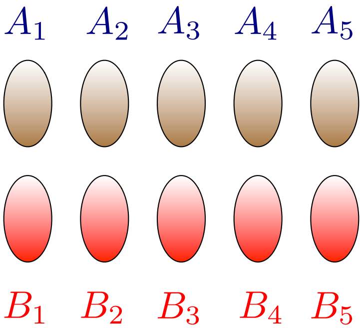
Here, $\{0\}$ denotes the subspace consisting of the zero vector only.

Two Families Theorem: Subspaces



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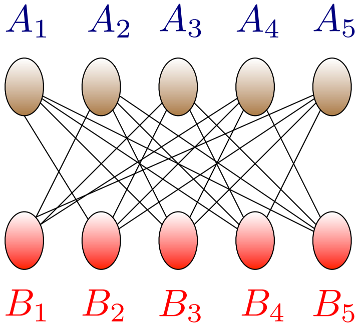

Draw an edge between two subspaces if the intersect!

Two Families Theorem (Subspaces): Lovász Theorem

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Two Families Theorem (Subspaces): Lovász Theorem

An useful reformulation:

Let M be a matrix of dimension $s \times n$ over $\mathbb{F}$. Furthermore, let $A_1, \ldots, A_m$ be p sized subset of columns such that each A_i are linearly independent and $B_1, \ldots, B_m$ be q sized subset of columns such that each B_j are linearly independent. Moreover,

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If $s = (p+q)$ then we can say,

$$\det[A_i \cup B_j] \neq 0 \iff i = j$$

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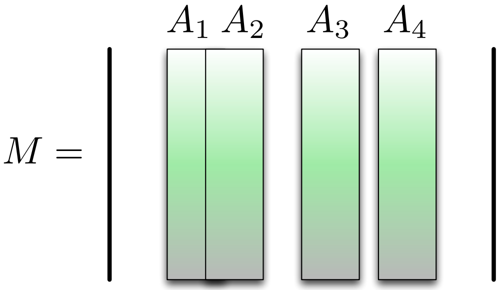

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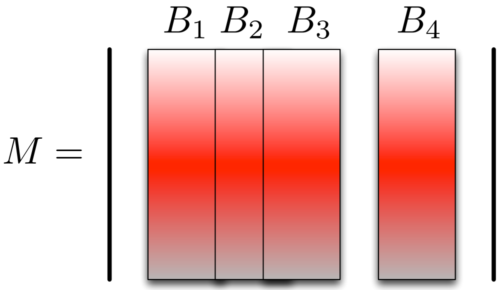

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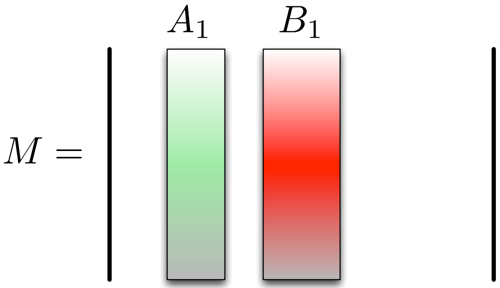

Two Families Theorem (Subspace)
Application III

Let G be a clique on n vertices and let $A_1, \ldots, A_m$ be forests on p edges and $B_1, \ldots, B_m$ be forests on $n-1-p$ edges such that $A_i \cup B_j$ is a *spanning tree* if and only if $i = j$.

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Can we say something better using Lovász Theorem?

Like $m \leqslant \binom{p+n-1-p}{p} = \binom{n-1}{p} \leqslant 2^n$!

Making our matrix!

Consider the matrix M with a row for each vertex $i \in V(G)$ and a column for each edge $e = ij \in E(G)$. In the column corresponding to $e = ij$, all entries are 0, except for a 1 in i or j.

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This is basically vertex-edge incidence graph of G. A set of edge X forms a forest in G if and only if columns corresponding to X are linearly independent in M over the finite field $\mathbb{F}_2$.

Proof?

- If G has a cycle then the corresponding columns adds up to 0?

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- If G has a cycle then the corresponding columns adds up to 0?

- Let X be a set of columns that are linearly dependent then the corresponding edges form a subgraph of even degree?

More Combinatorial Applications

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More Combinatorial Applications

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Read the two amazing surveys by
Zsolt Tuza
Applications of the Set Pair Method in Extremal Hypergraph Theory
Applications of the Set Pair Method in Extremal Problems, II
http://gilkalai.wordpress.com/2008/12/25/lovaszs-two-families-theorem/
http://www.thi.informatik.uni-frankfurt.de/ jukna/EC_Book_2nd/katona.html

Final Slide

Thank You!
Any Questions?